Estimating the Constant Elasticity of Substitution when Technical Change is Time-Varying: A Kalman Filtering Approach*

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Abstract

This paper presents a new approach to, simultaneously, provide an estimate of factor elasticities as well as time-varying technical change utilizing the Kalman filter. Using a simulation exercise we show that this approach performs well for different types of non-linear technical change. The method is subsequently applied on Danish macroeconomic data from 1966-2016 for the private sector and at the sectorial level. Using a nested CES production function we find that all inputs in production are gross complements. Specifically, our estimates suggest that the relevant elasticity of substitution between capital and labor is in the range of 0.6-0.9. Moreover, we find that technology in the private sector has been labor-augmenting in the long run, thus supporting an assumption of non-Hicks-neutral technical change. However, we find important "medium run"-fluctuations and periods where technical change has been capital-augmenting. Our analysis suggests that this result is driven by periods of relatively slow labor productivity growth in the service sector in the early 1990's and after the financial crisis.

Keywords: Biased Technical Change, Medium Run, Factor Substitution, Constant Elasticity of Substitution, Kalman Filter.

JEL: C32, E25, O33.

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1 Introduction

The elasticity of substitution between production factors is a central parameter in macroeconomic models. It affects the speed of output adjustment to shocks when factor prices are affected asymmetrically and for the induced price effects of tax or labor market reforms. Over time, it affects the distribution of income, for example between capital and labor as well as cross-sectional employment. In the presence of augmenting (so-called "biased") technical change, the elasticity of substitution determines how firms adjust demand for production factors in response to shifts in their relative efficiency.

Many empirical papers have estimated "the" elasticity of substitution, typically between labor and capital. Although a central parameter, no consensus value has been established, which may to a large extend be attributed to the identifying assumptions on technical change.[1] In a widely cited paper, Berndt (1976) assumes Hicks-neutral technical change and suggests a unitary value of this elasticity for the US economy, hence providing support for the use of a Cobb-Douglas production function. Later work, for example Antras (2004) and Leon-Ledesma et al. (2015), challenge this and show that the assumption of Hicks-neutrality will bias the estimated elasticity towards one when the true technical change is biased. More recently, it has been acknowledged that technical change, while labor-augmenting in the long run, might be characterised by prolonged periods where it is instead capital augmenting. To allow for dynamics during what Blachard (1997) has dubbed "the medium run", a series of papers (see for example Klump et al. (2007; 2008) and McAdam and Willman (2013)) depart from the linear trend assumption in earlier work by using a Box-Cox transformation of the growth rates to incorporate time-varying (non-linear) technical change. Although the Box-Cox transformation is able to get medium run fluctuations, it can only account for accelerating or decellerating growth rates. To get labor augmenting technology in the long run, while allowing for multiple periods of capital augmenting technology, a more flexible approach is needed.

We propose a framework to estimate the constant elasticity of substitution (CES) in a production function with time-varying technical change by using the fact that the problem has a natural state space representation. Thus, the Kalman filter allows us to simultaneously obtain an estimate of the CES elasticity as well as the (unobserved) relative augmenting technical change. We show how the identification is based on a smoothness restriction on technical change which replaces a full parametric specification of the time trends. We also show that our approach nests the constant growth assumption widely applied in empirical work. The approach is fairly easy to implement as an alternative to models with parametric assumptions about the technichal change.

The performance of our approach is assessed in a simulation exercise using three different

[1] See for example Knoblach and Zwerschke (2016) for a meta regression analysis using 738 estimates from 41 different studies.

specifications of technical change: One linear case and two nonlinear cases. We find that the Kalman filter performs well: It is able to recover the true elasticity on the median for all three trend specifications. Further, in the cases of non-linear technical change, the distribution of the estimates is much more narrowly centered around the true value, compared to a model which (incorrectly) use a linear trend assumption. Finally, the simulation excercise suggests that our approach is robust to misspecification with regards to the degree of smoothing.

Next, we apply the method to Danish aggregated and sectoral data for the period 1966-2016. We use a nested CES production function in which energy, capital (mashinery, transportation and inventory), labor, buildings and materials, respectively, enter as inputs in production.[2] To summarize our results, we find that all factors enter the production function with an elasticity on the unit interval, meaning that they are gross complements rather than substitutes. Specifically, our point estimates of the elasticity of substitution between the capital-energy nest and labor are in the range 0.6-0.9. We find moderate substitution between energy and capital, whereas buildings and materials are approximately Leontief factors in the production. These general conclusions hold both at the aggregate private sector level and for the two main sectors in Denmark (service and manufacturing). We find that technical change in the private sector has been labor-augmenting in the long run. However, we find that there has been periods with capital augmenting technical change for the private sector as a whole, e.g. in the 1990's and after the financial crisis in 2008. The former is in accordance with the results in Klump et al. (2008) who in part ascribes this structural shift to the IT boom. Interestingly, the latter coincides with a substantial increase in markups following the crisis. Overall, our findings support the assumption that technical change is labor augmenting in the long run, but that there are prolonged periods with capital augmenting technical change.

Since the true signal-to-noise ratio in the data is unknown this begs the question whether the chosen smoothness restriction of technical change (signal-to-noise ratio) is appropriate. The choice of benchmark signal-to-noise ratio is in the model specification motivated from an economic perspective and in the empirical application from an econometric perspective using several statistical tests. The level of smoothing that we choose is appropriate across sectors and factor nests, which is not the case if changed substantially in either direction. We find that the estimated elasticities are generally robust to moderate variations in the degree of smoothness around our benchmark specification. However, we do see that the elasticity estimates are small for very low degrees of smoothing and vice versa.[3] Although we are convinced that our choice of signal-to-noise ratio is

[2]The nesting structure is based on the one used in the Danish macroeconomic model MAKRO and does not affect the methodological contribution of this paper.

[3]Thus, this paper reiterate the findings in Chirinko and Mallick (2017), that the elasticity is bound to the filtering assumptions made. However, while their filtering assumptions are made on the smoothing of prices prior to estimation, we smooth technical change as part of the estimation.

appropriate, a more detailed methodology to determine the smoothness restriction in production technology would be interesting. We leave this question for future research.

The remainder of this paper is structured as follows: In Section 2 we take a closer look at some stylized facts for the Danish economy and motivate the use of time-varying technical change. We present the CES production function and nest structures that we estimate in Section 3 along with a presentation of the econometric framework. A simulation exercise is conducted in Section 4 and applied on Danish aggregate data in Section 5, where we also perform a range of diagnostics tests and analyze the appropriateness of the smoothness restriction. Section 6 concludes.

2 The case for time-varying technical change: A look at Danish data

The empirical litterature commonly uses an assumption that technical change grows at a linear rate, typically with a further a priori assumption that this growth is either Hicks- or Harrod-neutral. While this might be an accurate description of the economy's long run (or steady state or balanced growth path), in this section we will argue that there are prolonged periods for Denmark that deviate from this description. Further, we will argue that it is necessary to allow the econometric model to reflect these medium run properties of the data when estimating the elasticity of substitution.

As pointed out by Blachard (1997), the assumption that the medium run is merily a transition period between the short run, typically defined as business cycles, and the long run, characterized by a steady state, is misleading. The panels in Figure 1 show in different ways that such a description of the medium run is incorrect for Denmark as well: There are visible changes and patterns that are more persistent than the typical business cycle length (2-8 years) but do not describe a balanced growth path neither. For example, panel a) shows the expenditure shares on the different production factors in the Danish private sector as a whole (see Section 5.1 for a detailed explanation of the data series used). If the production function was Cobb-Douglas (unitary elasticity of substitution) the expenditure shares would be constant and the direction of technical change would not affect factor income. However, we do observe an increasing share of (machinery) capital and materials, while the opposite is true for labor and buildings. This is even more clear in panel b) where we consider only capital and labor for the two major private sectors in the Danish economy: Manufacturing and private services. The panel shows the income share of capital, which is clearly non-constant, again contradicting a Cobb-Douglas specification of the production function. This is especially true for manufacturing, which has seen an upward trend in the capital income share from 18% in 1986 to 32% in 2015. A similar rise is seen in several other

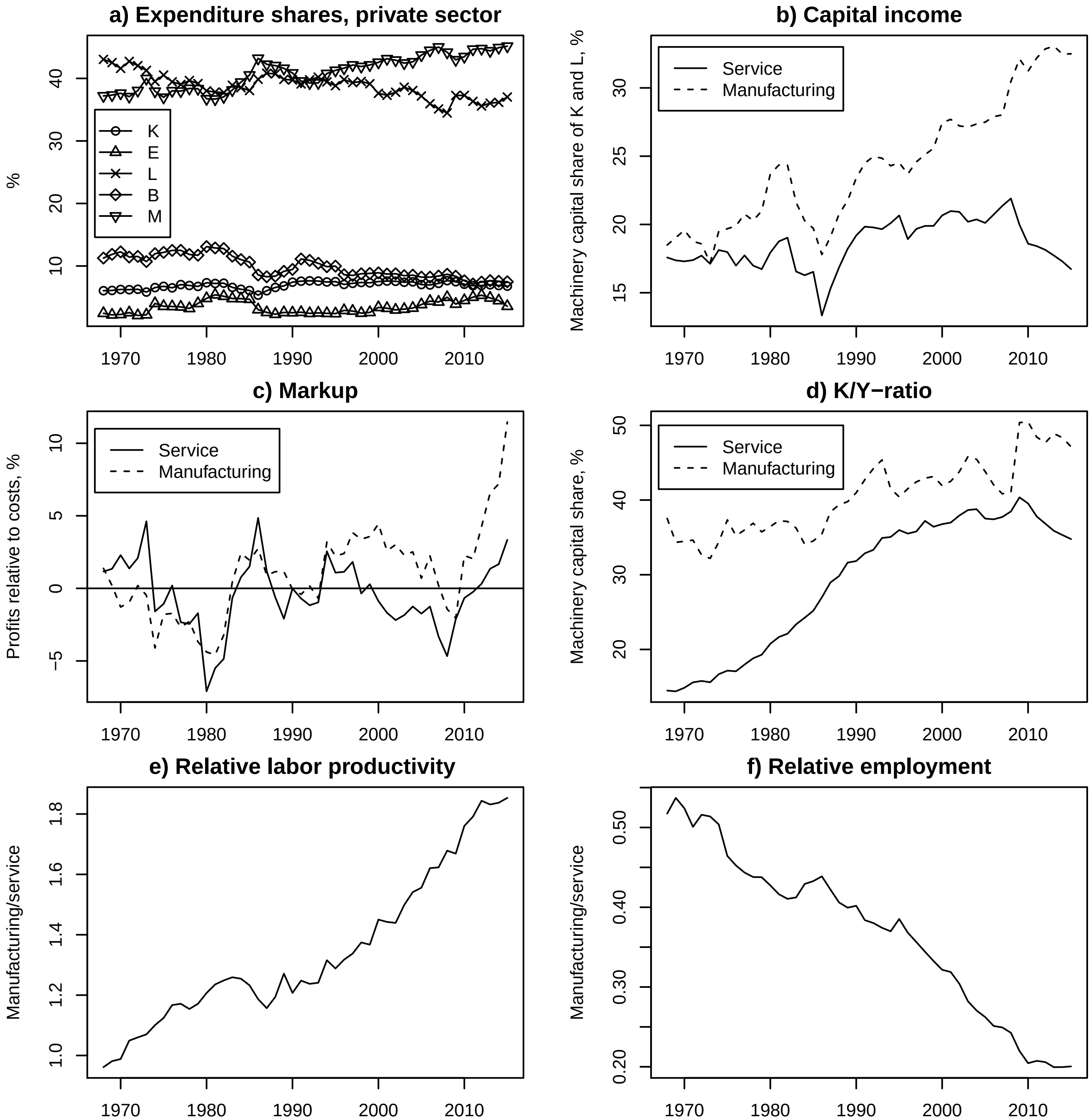

Figure 1: Aggregate Danish data on factor components. a) Expenditure shares in the private sector. b) Capital income share calculated as $\frac{p_t^K K_t}{w_t L_t + p_t^K K_t}$, where p_t^K is a usercost of machinery capital. c) Markup in service and manufacturing. d) K/Y-ratio calculated as machinery capital relative to total production in service and manufacturing. e) Labor productivity calculated as $\frac{Y_t}{L_t}$, i.e. the "value-added" for every working hour. Manufacturing relative to private services. f) Number of employed in manufacturing relative to services.

European economies as well, see e.g. Arpaia et al. (2009) and Karabarbounis and Neiman (2014). The latter find that around 50% of the decrease in the labor share can be subscribed to changes in the relative price, highlighting that the change in factor shares is not only driven by the relative prices. For services it might be argued that the long run capital income share is constant but with important medium run variation (the capital income share troughs at 13% in 1986 and peaks at 22% in 2008). McAdam and Willman (2013) and Blachard (1997) argue that this is likely reflecting biased and time-varying technical change and its interplay with long adjustment lags to the K/L-ratio as well as so-called directed technical change, where profits determine the level of research and development level in different sectors and across different factors of production.[4] In panel c) we see that, while somewhat noisy, both markup series show considerable persistency over time.[5] The increase in markups after the financial crisis is particularly pronounced for Denmark but echoed in several other economies (De Loecker and Eeckhout (2018)). Interestingly, periods of high markups seem to preceede periods of decreasing capital income shares (and vice versa) suggesting that high markups could drive directed investment in research and development. The rejection of the "stylized fact" of a constant K/Y-ratio in panel d) is another case in point (we consider here again only machinery capital). The series exhibit hump-shapes, especially pronounced in the service sector. The period of rapidly increasing use of capital in the production followed by less rapid growth is hard to reconcile with a constant rate of technical change as we shall see. Finally, panel e) and f) illustrate some significant intersectorial shifts between manufacturing and services during the sample period. The relative productivity shows a clear trend with labor productivity growth in manufacturing exceeding that in private services by 1.4% per year on average. Inversely, the relative employment in the two sectors exhibit a clear downward pattern in the same period. This pattern of labor saving technology, where the excess labor in manufacturing finds employment in the service sector is consistent with labor augmenting technical change and an elasticity of substitution between labor and other relevant production input of less than unity (which we do in fact find in our estimation). However, this trend has not been constant but accelerated during the 2000s (the relative productivity was increasing by 1.8% yearly from 1968-1983, decreasing by 2.1% from 1983-1987, increasing by 1.5% from 1987-2002 and then accelerating to 2.5% from 2002-2012).

To conclude, factor demand and income, markups and productivity in Denmark clearly show persistent variation that can be characterized as medium run properties which will be hard to reconcile with a linear trend model. We follow the litterature that explains (at least partly) these properties by biased technical changes that do not happen at a constant rate but are instead time-varying. These changes may be caused by a number of factors such as competitive structure, labor

[4]See for an example Acemoglu (2002; 2003) for a theoretical model with endogenous innovations leading to directed technical change.

[5]We calculate the markup as profits relative to expenditures. This approximation only holds if there is constant returns to scale in production, which we assume for simplicity.

market institutions, demographics, etc., which affect factor demand in ways that are unknown to the econometrician. This motivates the model presented in the next section where technical change is allowed to grow non-linearly while still being specified in a flexible, data driven way.

3 Model specification

We assume that the firm produces in accordance with a nested CES production function and substitute between two factors in each nest. This specification has the advantage that it allows for different elasticities of substitution between the factors in the production process. First, the producer chooses between capital (K) and energy (E). Secondly, the producer chooses between the nest of K and E (KE) against labor (L). Thirdly, the producer chooses how much to spent on buildings (B) and the production input consisting of capital, energy and labor (KEL). Lastly, the producer chooses how much to produce itself (KELB) and how many materials to buy from other producers (M). In short, we assume that the firm produces in a KELBM structure, which is the notation we will use in the rest of the paper. The production in each nest is given by:

$$Y_t = \left[(\Gamma_{1t} X_{1t})^{\frac{\sigma-1}{\sigma}} + (\Gamma_{2t} X_{2t})^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}, \tag{1}$$

where Y_t is output, X_{1t} and X_{2t} are two production factors in a given nest, and σ is the constant elasticity of substitution. Γ_{1t} and Γ_{2t} represents augmenting technical change of the first and second factor, respectively. From now on we denote the ratio $\frac{\Gamma_{1t}}{\Gamma_{2t}}$ as the relative augmenting technical change of the two factors of production and it is also this ratio that we plot in the empirical analysis. Let $g(\Gamma_{1t})$ and $g(\Gamma_{2t})$ be the growth rates of the factor augmenting technologies. If $g(\Gamma_{1t}) > g(\Gamma_{2t})$ technical change is augmenting the first production factor, whereas $g(\Gamma_{1t}) < g(\Gamma_{2t})$ implies that technical change is augmenting the second factor. Assuming profit maximizing firms we can derive the relative expenditure shares for the two production factors derived from the firms cost minimization problem as:[6]

$$log\left(\frac{P_{1t}X_{1t}}{P_{2t}X_{2t}}\right) = (\sigma - 1)\, log\left(\frac{\Gamma_{1t}}{\Gamma_{2t}}\right) + (1-\sigma)\, log\left(\frac{P_{1t}}{P_{2t}}\right), \tag{2}$$

where P_{1t} and P_{2t} are the prices of the two factors. Equation (2) illustrates how biased technical change and changes in the relative price interact with the elasticity of substitution: When $\sigma < 1$ the production factors are gross complements and when $\sigma > 1$ they are gross substitutes. Two special cases emerge when $\sigma \to 1$ where production is Cobb-Douglas, resulting in constant relative expenditure shares, and $\sigma = 0$ where production is Leontief and there is no substitution between the

[6] Using this formulation has the advantage that it is less plausible that there are correlated measurement errors on both the right and left hand side, which could bias the elasticity estimate (Kemp, 1962).

production factors due to price changes (perfect complements). Note that Hicks neutral technical change, affecting both technology factors in the same way, will not affect the *relative* technology level in (2) and hence does not affect the *relative* expenditure shares. Thus, we are only able to identify biased technical change in the relative series. As an example, consider the second nest with KE as the first factor and L as the second factor: If $g(\Gamma_{KEt}) > g(\Gamma_{Lt})$, technical change is augmenting KE. If $\sigma < 1$, technical change is biased towards L, whereas technical change is biased towards KE when $\sigma > 1$. This highlights the well-known result that the effect of biased technical change depends on the elasticity of substitution and therefore, they have to be estimated jointly.

3.1 Obtaining the state space representation of the model

In order to apply the Kalman filter we specify the model as a linear state space model. Thus, we have to specify both an observation and a state equation. Equation (2) is a static equation as it does not contain any dynamics. Thus, it is explicitly assumed that the economy is in a long run equilibrium and is the static long run solution from the Engle-Granger two-step procedure. However, in macroeconomic modeling, adjustment costs are often imposed to ensure that there are lags in the response of quantities to relative price changes (e.g. Christiano et al., 2005 and Smets and Wouters, 2007). Consequently, using (2) might result in a small sample bias due to the relatively short sample that we use in our application. Therefore, we employ an error-correction model to allow for short run dynamics in factor shares.[7] Specifically, we estimate the equation:

$$\Delta s_t = \alpha\left(s_{t-1} - \beta p_{t-1} - \mu_{t-1}\right) + \sum_{i=0}^{k} \kappa_i \Delta p_{t-i} + \sum_{i=1}^{k} \omega_i \Delta s_{t-i} + \varepsilon_t, \qquad \varepsilon_t \sim N(0, \Sigma^{\varepsilon}) \tag{3}$$

where $s_t \equiv log\left(\frac{P_{1t}X_{1t}}{P_{2t}X_{2t}}\right)$ is the relative expenditure share, $p_t \equiv log\left(\frac{P_{1t}}{P_{2t}}\right)$ is the relative (log)prices and $\mu_t \equiv (\sigma - 1)\, log\left(\frac{\Gamma_{1t}}{\Gamma_{2t}}\right)$ captures the relative technology level. The speed of adjustment to the long run equilibrium is determined by α, $\sigma = \beta + 1$ is the long run elasticity and κ_i and ω_i are the short run elasticities with respect to prices and expenditure shares, respectively. The optimal lag length (k) is determined such that there is no autocorrelation in the measurement errors, ε_t. The variance Σ^{ε} is estimated by a recursively application of the maximum likelihood estimator.

Next, we need to specify the dynamics of μ_t to satisfy three requirements: First, since the relative factor price contains a trend in many of the estimations, we need to allow for a trend (implicitly to allow for $\sigma \neq 1$). Second, we deviate from the linear trend assumption to account for the medium-run fluctuations in factor expenditures. Third, we require it to be a slow moving process, meaning it cannot reflect the year-to-year errors between the data and the model. This

[7] As first seen in Caballero (1994) and later used in Bonga-Bonga (2009)

last characteristic is indeed how we define and identify movements in factor efficiency.[8] Specifically, we assume the following I(2)-process for μ_t:

$$\Delta\mu_t = \Delta\mu_{t-1} + \eta_t, \qquad \eta_t \sim N(0, \Sigma^\eta). \tag{4}$$

To control for the smoothness of technical change we restrict the variance of the state errors η_t to $\Sigma^\eta = \Sigma^\varepsilon/\lambda$, where λ (the inverse of the signal-to-noise ratio) determines how aggressively fluctuations should be smoothed. We choose a moderate value of the signal-to-noise ratio in our benchmark specification which we consider reasonable from an economic perspective: Extremely low degrees of smoothing (a high signal-to-noise ratio) would ascribe almost the entirety of year-to-year fluctuations in factor demand to changes in technology[9], whereas extremely high degrees of smoothing imposes a linear trend assumption on the model, something we find unlikely as motivated in section 2.[10] In section 5.4 we also motivate the choice of signal-to-noise ratio from an econometric standpoint and show that our benchmark value is appropriate. The dependency of the signal-to-noise ratio illustrates that the state space model nests some widely different interpretations of what constitutes the "structural" demand for a given input factor. It also highlights that, while we do not need to impose a functional form of technical change, we do need to make an identifying assumption based on the smoothness of the structural level. We set $\lambda = 100$ in our benchmark specification, reminiscent of typical applications of the Hodrick-Prescott filter. As a result, ε_t captures short run (temporary) deviations from the long run equilibrium whereas η_t captures structural changes in the relative augmenting technical change.

As the Kalman filter can be sensitive to starting values in small samples 10 different values of the elasticity parameter σ and the adjustment parameter α are tried (i.e. a grid of a total of 100 combinations of initial parameter values). The starting values that minimize the AIC are chosen. However, for most nests we find the estimation procedure to be relatively stable, meaning that it converges to the same optimum for most of the starting values. In addition, we have restricted the point estimate of the elasticity to be non-negative. The elasticity and adjustment parameter are specified as unobservables with a zero variance in the state space representation. Therefore, we do not obtain a standard error by using the Kalman smoother. Instead, we use a standard residual-based fixed-design bootstrapping procedure with 1,000 iterations to obtain confidence intervals, which is valid given that the model's innovations are neither autocorrelated nor heteroskedastic (we will verify this in the empirical application in Section 5).[11] We report the confidence bands by the

[8] Previous empirical work on Danish data has generalized this assumption by using higher-order polynomials (Gustafsson, 2014) or logistic functions (Thomsen, 2015).

[9] This is often referred to as "dynamic calibration" in the realm of CGE-models where parameters are calibrated each year to match the data set.

[10] Thus, our "smoothness-restriction" is closely related to the periodicity in the low-pass filter used in Chirinko and Mallick (2017) prior to estimation.

[11] One could also consider a Monte Carlo procedure to obtain confidence intervals as described in Hamilton (1986).

2.5% and 97.5% quantiles instead of the standard errors, as the distribution of the bootstrapped parameters might be non-normal and not necessarily centered around the estimated parameter values.[12]

4 Simulation evidence

As illustrated in Section 2, technical change in Denmark is not likely to be linear, thus violating the common linear trend assumption. In this section, we perform a simple simulation study to show that in such cases, the Kalman filter will provide a more accurate measurement of the elasticity of substitution than the linear trend assumption. Specifically, we analyze the factor shares of two generic factors and try to match the data generating process (DGP) with the moments of capital and labor as well as the expenditure share observed in the datasets used in the empirical analysis. Appendix A includes a detailed description of how the data is simulated.

Technology is specified such that it consists of a deterministic and a stochastic trend. We vary the relative importance of these by adjusting the signal-to-noise ratio ($\tilde{\lambda}^{-1}$) in the DGP while keeping the total variance constant. Three different specifications of the deterministic trend are considered: The first is the constant Harrod-neutral growth rate assumption. The second specification is also a constant growth rate with a break in the augmenting technology, i.e. we assume Harrod-neutrality in the first half of the sample and Solow-neutrality in the second half. The last specification uses a Box-Cox transformation of the growth rates to account for medium run fluctuations. The parameters are specified such that the growth rate of labor productivity is declining, whereas the growth rate of capital productivity is increasing, similar to what might be observed empirically. We simulate 1,000 data series of 50 observations, consistent with what is used in the empirical analysis.

The estimation methodology is as described in Section 3.1. Importantly, we fix the smoothing parameter in the Kalman filter at the benchmark value ($\lambda = 100$) irrespectively of the true signal-to-noise ratio in the DGP. The results of the first specification are shown in Table 1. On the median, both the linear trend and the Kalman filter suceed in replicating the true elasticities. However, when $\tilde{\lambda}$ is low (implying that the stochastic component in the technology process is dominant) the precision of the Kalman filter is far better than the linear trend assumption based on the distance between the quantiles. When technology approaches a linear trend (i.e. when $\tilde{\lambda}$ is high) the linear trend performs slightly better than the Kalman filter.

However, this procedure only works when λ is estimated freely, as the absolute noise level in the filter does not affect the estimates.

[12] We restrict the parameter estimates of σ in the bootstrapping procedure to be in the range between -0.75 and 1.75 to discard extreme outliers, likely due to numerical convergence problems. The "acceptance ratio" obtained was fairly close to 1 anyway.

	$\sigma = 0.2$	$\sigma = 0.5$	$\sigma = 0.9$	$\sigma = 1.3$
		$\tilde{\lambda} = 1$		
Linear trend	0.19	0.49	0.89	1.30
	$(-0.23;0.68)$	$(-0.14;1.19)$	(0.14;1.69)	(0.57;2.03)
Kalman smoother	0.20	0.50	0.90	1.30
	$(-0.04;0.49)$	(0.14;0.91)	(0.48;1.37)	(0.89;1.70)
		$\tilde{\lambda} = 10$		
Linear trend	0.20	0.50	0.90	1.30
	(0.04;0.37)	(0.27;0.75)	(0.64;1.18)	(1.05;1.58)
Kalman smoother	0.20	0.50	0.90	1.30
	(0.07;0.34)	(0.31;0.70)	(0.67;1.13)	(1.09;1.51)
		$\tilde{\lambda} = 100$		
Linear trend	0.20	0.50	0.90	1.30
	(0.12;0.29)	(0.38;0.63)	(0.76;1.05)	(1.17;1.44)
Kalman smoother	0.20	0.50	0.90	1.30
	(0.09;0.31)	(0.35;0.65)	(0.72;1.08)	(1.14;1.46)
		$\tilde{\lambda} = 1000$		
Linear trend	0.20	0.50	0.90	1.30
	(0.13;0.28)	(0.40;0.61)	(0.78;1.03)	(1.19;1.42)
Kalman smoother	0.20	0.50	0.90	1.30
	(0.10;0.30)	(0.35;0.65)	(0.73;1.08)	(1.14;1.46)

Table 1: Estimated (median) elasticities on the simulated data using a constant growth rate specification with Harrod-neutral technology. 1,000 simulations are used and 50 observations. 5% and 95% quantiles are included in paranthesis. $\tilde{\lambda}$ is the inverse of the signal-to-noise ratio used in the data generating process.

	σ= 0.2	σ= 0.5	σ= 0.9	σ= 1.3
	$\tilde{\lambda} = 1$			
Linear trend	0.29 (−3.03;4.58)	0.52 (−0.79;1.79)	0.89 (0.14;1.68)	1.29 (0.32;2.22)
Kalman smoother	0.20 (−0.15;0.57)	0.50 (0.12;0.93)	0.90 (0.48;1.37)	1.31 (0.87;1.72)
	$\tilde{\lambda} = 10$			
Linear trend	0.26 (−2.63;3.05)	0.54 (−0.22;1.27)	0.90 (0.61;1.22)	1.29 (0.83;1.78)
Kalman smoother	0.21 (−0.05;0.49)	0.50 (0.29;0.73)	0.90 (0.68;1.13)	1.30 (1.08;1.51)
	$\tilde{\lambda} = 100$			
Linear trend	0.27 (−2.95;3.38)	0.53 (−0.11;1.17)	0.90 (0.70;1.10)	1.29 (0.91;1.70)
Kalman smoother	0.21 (−0.02;0.46)	0.50 (0.33;0.68)	0.90 (0.72;1.08)	1.30 (1.13;1.48)
	$\tilde{\lambda} = 1000$			
Linear trend	0.27 (−2.49;3.09)	0.54 (−0.12;1.15)	0.91 (0.73;1.08)	1.29 (0.91;1.69)
Kalman smoother	0.20 (−0.02;0.45)	0.51 (0.33;0.67)	0.90 (0.73;1.07)	1.30 (1.13;1.47)

Table 2: Estimated (median) elasticities on the simulated data using a constant growth rate specification with Harrod-neutral technology in the first half of the sample and Solow-neutral in the second half. 1,000 simulations are used and 50 observations. 5% and 95% quantiles are included in paranthesis. $\tilde{\lambda}$ is the inverse of the signal-to-noise ratio used in the data generating process.

	σ= 0.2	σ= 0.5	σ= 0.9	σ= 1.3
	$\tilde{\lambda} = 1$			
Linear trend	0.21 (-0.69;1.03)	0.50 (-0.31;1.28)	0.89 (0.12;1.70)	1.31 (0.51;2.10)
Kalman smoother	0.20 (-0.05;0.49)	0.50 (0.14;0.91)	0.90 (0.48;1.37)	1.30 (0.88;1.71)
	$\tilde{\lambda} = 10$			
Linear trend	0.22 (-0.29;0.73)	0.50 (0.13;0.90)	0.90 (0.63;1.20)	1.31 (0.99;1.64)
Kalman smoother	0.20 (0.06;0.34)	0.50 (0.31;0.70)	0.90 (0.67;1.13)	1.30 (1.09;1.51)
	$\tilde{\lambda} = 100$			
Linear trend	0.22 (-0.23;0.67)	0.51 (0.19;0.81)	0.90 (0.74;1.06)	1.30 (1.08;1.52)
Kalman smoother	0.20 (0.09;0.32)	0.50 (0.35;0.66)	0.90 (0.72;1.08)	1.30 (1.13;1.46)
	$\tilde{\lambda} = 1000$			
Linear trend	0.22 (-0.22;0.66)	0.51 (0.22;0.79)	0.90 (0.77;1.04)	1.30 (1.09;1.51)
Kalman smoother	0.20 (0.09;0.31)	0.50 (0.35;0.65)	0.90 (0.73;1.07)	1.30 (1.14;1.46)

Table 3: Estimated (median) elasticities on the simulated data using a Box-Cox transformation of the growth rates of technology. 1,000 simulations are used and 50 observations. 5% and 95% quantiles are included in paranthesis. $\tilde{\lambda}$ is the inverse of the signal-to-noise ratio used in the data generating process.

Next, consider the second specification with a break after 25 observations. The results are shown in Table 2. In this case, the linear trend assumption is far less accurate on the median than the Kalman filter in all specifications, reflected in wider confidence bands. The difference is most pronounced when the true elasticity is furthest away from a Cobb-Douglas productions function. This is of course not particularly surprising but highlights the fact that a misspecified process for technical change most severely (adversely) affects the estimates when the effect of technical change is large (σ being low).

Finally, we employ a Box-Cox transformation of the growth rates. The results are shown in Table 3. Again we find that the Kalman filter performs better than the linear trend on the median and on the width of the confidence bands - in particular when $\tilde{\lambda}$ is low. To conclude, we have shown that the Kalman filter performs almost as accurate as the linear trend assumption when technology is close to linear, but also that the Kalman filter is far more accurate when technology is non-linear. Although the signal-to-noise ratio used in the Kalman filter is not identical to the true ratio in the simulated data we still get fairly precise estimates. Being convinced of the flexibility and robustness of our method, we move on to an empirical application on Danish data.

5 Empirical application

In this section, we apply the framework presented in Section 2 on Danish data. In Section 5.1 we describe how the data is generated and our data source. In Section 5.2 we present the main results. In Section 5.3 we analyze the filter performance and consistency. Lastly, Section 5.4 includes a detailed analysis and discussion of the validity of the identification strategy.

5.1 Data

The data source is Statistics Denmark from which we obtain annual time series for the period 1966-2016. We use data from the private sector as a whole (leaving out the public sector), as well as the manufacturing and private services sectors. The variables are defined analogous for all sectors used. As a measure of capital we use two types, which we separate: Machinery, transportation and inventory capital and building capital. The user cost of capital is calculated as shown in Appendix B. This usercost is used to calculate the price of both types of capital. Specifically, a "steady state" user cost (in the absence of installation costs) and a user cost with adaptive expectations are calculated (which includes installation costs). The steady state user cost is used in the long run relation of (3) and the user cost with adaptive expectations is used in the short run relation. This distinction mimics typical specifications used in DSGE models, where adjustment of the capital stock is restricted by some cost function in the short run, while this cost is usually zero in the steady state. As a measure of demand for energy we used the quantity spent on energy and the measure of materials is given as all materials used in production excluding energy. Number of working hours are used as a measure of labor input and the price of labor is the hourly wage.

The price indices in the nests KE, KEL and KELB are calculated as a Paasche index. This index has the advantage that it is close to the CES price index. The quantities are derived through a zero profit condition. In the last nest (the nest between KELB and M), the production volume is used as measure of output and the price is given as the price of total production adjusted by a markup. In this way, the zero profit condition holds given an optimizing price. The time series used for estimation are shown in Figures 3-5 (see Appendix C).

5.2 Results

In this section, we present our results using the Danish datasets. The estimated elasticities for the private sector are shown in Table 4 and the relative augmenting technical change $\left(\frac{\Gamma_{1t}}{\Gamma_{2t}}\right)$ is shown

	(K)E	(KE)L	(KEL)B	(KELB)M
σ	0.19	0.69	0.16	0.00
	(0.06;0.45)	(0.26;1.58)	(−0.22;0.94)	(−0.54;0.67)
α	−0.41	−0.13	−0.23	−0.36
	(−0.53;−0.20)	(−0.23;−0.05)	(−0.35;−0.05)	(−0.52;−0.12)
k	0	0	0	0
Likelihood	98.10	132.55	128.67	124.85
R^2	0.95	0.98	0.94	0.72
Autocorrelation	[0.97]	[0.77]	[0.09]	[0.13]
Heteroskedasticity	[0.74]	[0.28]	[0.05]	[0.35]
Normality	[0.50]	[0.88]	[0.78]	[0.47]
NIS	0.84	0.76	0.83	0.85

Table 4: Private sector: Estimated results in a (((KE)L)B)M nest structure. Terms in parenthesis are the lower and upper critical values, respectively, on a 5% significance level. Terms in brackets are p-values for the misspecification tests. The critical values of the Normalized Innovations Squared test (NIS) are $[0.63; 1.45]$ on a 5% significance level.

in Figure 2. The elasticity of substition between K and E is estimated to 0.19, but different from zero and unity. The elasticity of substitution between KE and L is found to be relatively high at 0.69, which can be explained by the fact that the capital stock is very persistent. The estimate is associated with some uncertainty as the distance between the quantiles is quite large. Consequently, we cannot reject that it is different from unity. The elasticity of substitution between KEL and B is found to be small (0.16) and not significantly different from zero. Therefore, we find that it is weakly identified from an economic perspective. It cannot be rejected that the estimated elasticity of substitution between KELB and M is zero, but it is different from unity. Thus, we find based on the point estimates that buildings and materials might be a Leontief input in production.[13] The detailed results for the manufacturing sector and private services are shown in Appendix D. Importantly, we find that the point estimate of the elasticity of substitution between KE and L is in the range 0.6-0.9.

Is these results consistent with the literature? Thomsen (2015) estimate the Danish elasticity of substitution between KE and L in the range 0.32-0.63 in the same sectors that we use. Gustafsson (2014) estimate the elasticity between K and L to 0.25 in manufacturing and 0.33 in services. Thus, our estimates tend to be higher than the literature on Danish data. Most importantly though our estimates imply the same bias of technical change. Klump et al. (2008) and McAdam and Willman (2013) estimate the elasticity for the Euro-area to be in the range 0.6-0.9, exactly as we do. Thus, we feel convinced that our estimates are not that far off what should be expected.

Next we turn to the direction of technical change. Since our estimates of the elasticity of

[13]This is in accordance with Gustafsson (2014), who restrictes both the substitution elasticity between KLE and B and the elasticity between KLEB and M to zero.

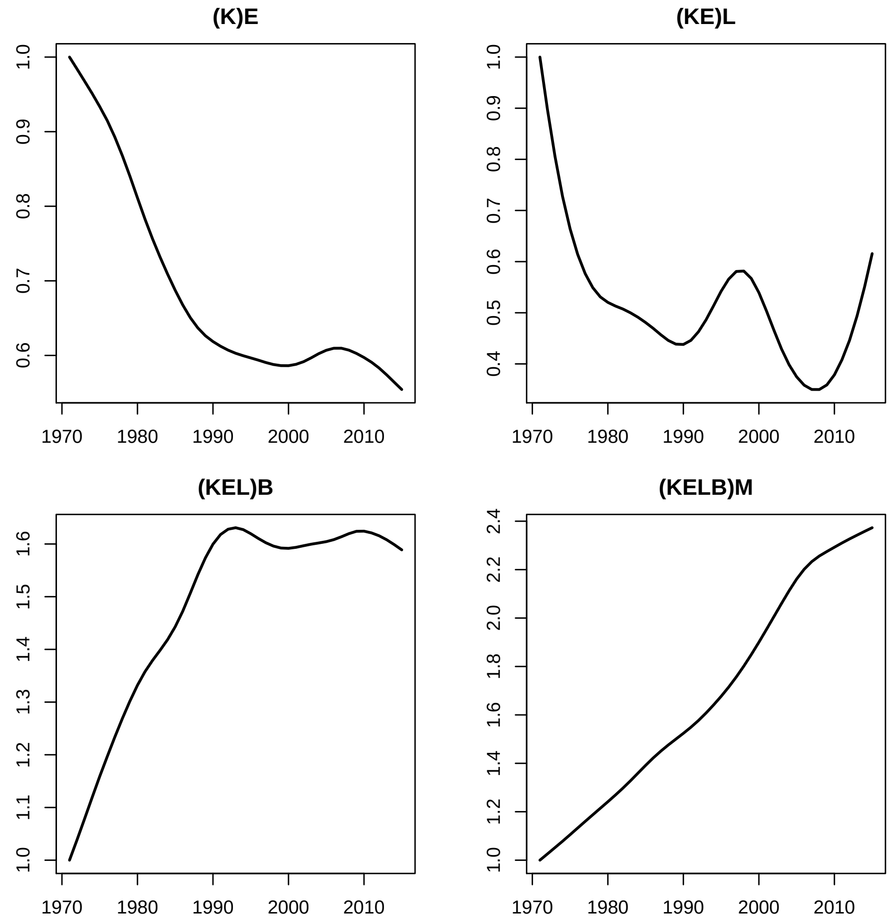

Figure 2: Private sector: Relative augmenting technical change in a (((KE)L)B)M nest structure. Paranthesis indicates the nested factors. That is, the graph shows the relative augmenting technologies of e.g. capital relative to energy. Index year is 1971.

substitution are all below unity, the bias of technical change goes in the opposite direction as the augmenting technology. For all sectors, the augmenting technical change of capital relative to energy has been decreasing (Figure 2 is the private sector, Figure 6 is manufacturing and Figure 7 is private services both shown in Appendix E). Likewise, for all sectors, the augmenting technical change of KE relative to L has been labor augmenting in the long run, but biased towards KE. However, there are important medium run fluctuations as the relative augmenting technical change in the private sector was increasing in the 90s and again in the end of the sample period where investments are known to be unusually low after the financial crisis.

In manufacturing, technical change was capital augmenting in the 80'es, possibly due to an increase in unemployment (Blachard, 1997). In the service sector, technical change was capital augmenting in the 90'es, possibly due to the IT boom (McAdam and Willman, 2013)[14], expected

[14]McAdam and Willman (2013) points out that a non-constant growth of technical change is a key component

to influence service the most. For both manufacturing and services we observe a change in the direction of technical change after the financial crisis. This might be due to low investments and increasing markups.

The augmenting technical change of KEL relative to B has been steadily increasing with a linear trend until 1990, where a slow-down is observed, possibly due to a declining growth in labor productivity. Lastly, the augmenting technical change of KELB against M has been increasing with a linear trend during the whole sample period implying that technical change is materials biased. In general we observe that our method allows for several periods of medium run fluctuations, declining relative growth rates and an approximately linear trend. These results illustrate the potential gains from the flexibility of our setup which would arguably not be obtained by an a priori parametric restriction on technical change.

5.3 Filter performance and consistency

It is an underlying assumption that ε_t and η_t are serially uncorrelated, unbiased and normally distributed. Under these assumptions the standardized innovations (the standardized forecast errors) are also serially uncorrelated, have a constant variance and are normally distributed. We test how well our model satisfies these assumptions through a series of statistical tests and later in a graphical analysis. The p-values of the tests are shown in Table 4 and Table 7-8 in Appendix F. Autocorrelation is tested based on an Breusch-Godfrey test of order 1. Generally, it cannot be rejected that the innovations are serially uncorrelated at the 5% significance level, implying that the model succeeds in accounting for all past dynamics influencing the market share.[15] Notably, in all nests in the private sector, zero lags are necessary to avoid autocorrelation. In all nests, the null of homoscedasticity cannot be rejected on a 5% significance level based on a Breusch-Pagan test for heteroscedasticity. In the nests where no autocorrelation and no heteroscedasticity are found, the Kalman filter recursions are known to be the optimal linear projection. Normality of the innovations is tested by applying a Jarque-Bera test for normality, and the null of normality cannot be rejected for all nests in the private sector.[16]

The filter performance and consistency can be evaluated graphically by inspecting the standardized innovations (these are shown in Figures 8-10 in Appendix F). If the filter behaves consistently, approximately 5% of the standardized innovations fall outside the 95% confidence intervals. Fur-

at understanding Euro-area medium run phenomena.

[15] In order to get a well calibrated filter in the manufacturing and private services, we had to remove outlier observations in the estimation between KEL and B. Concretely, we set observation number 1,5 and 39 in the manufacturing sector as NA and observation 5,6,24,39 to NA in the service sector. This changes the elasticity estimate only marginally, but have large effects on the misspecification of the model.

[16] There are signs of non-normality in some of the nests in the manufacturing sector but as noted in Hamilton (1994), even when the innovations are non-normal the Kalman filter can still be used to calculate a forecast of Δs_t and will still produce consistent and asymptotically normal estimates.

ther, they appear to be drawn from a zero mean and serially uncorrelated distribution, confirming the LM-test for autocorrelation. We also performed a more formal test for unbiasedness and misspecification of the overall noise level, using the Normalized Innovations Squared (NIS - see Appendix F for a detailed description). The test statistics for the private sector are shown in Table 4. For all nests the null hypothesis of unit mean cannot be rejected. We do find that, while within the confidence bands, the test statistic is slightly below one which could indicate that the overall noise level might be too high. However, the fact that we do not find prevailing autocorrelation in our model indicates that the Kalman gain is not suppressed excessively since the state is allowed to respond to the forecast errors in the updating step of the filter.

5.4 Validation of the identification strategy

As mentioned previously, the level of smoothness of factor efficiencies in the resulting state space model is determined by the signal-to-noise ratio (the inverse of the parameter λ). Not surprisingly, the estimate of σ depends on the signal-to-noise ratio, analogous to Klump et al. (2007) who report widely different estimates, depending on the trend assumption. To illustrate this, the estimated elasticities for different values of λ for the private sector are shown in Table 5 and Table 9-10 in Appendix F is the manufacturing and private services sectors, respectively. With few exceptions, we find that the estimated elasticities across sectors and nests are increasing in the level of smoothness. For example, in the KEL nest in the private sector in Table 5 the estimated elasticity is 0.05 when almost no smoothing is imposed on the model ($\lambda = 1$) and increasing to 0.99 when technical change follows an approximate linear trend. This result is closely related to the one found in Chirinko and Mallick (2017), where the elasticity of substitution was increasing for higher values of the periodicity parameter.[17] This highlights two important points: First, in line with the literature, our findings reiterate the tight link between identifying restrictions on technical change and the elasticity of substitution. Although our model specification is less parametric and seemingly less a priori restrictive we do not escape this fact. Second, since our identification strategy is based on the signal-to-noise ratio in our filter, we can check whether a particular smoothness level results in filter consistency and the absence of autocorrelated residuals. In this subsection we argue based on a set of statistical tests, that our identification strategy performs well.

Since the Kalman filter tends to produce serially correlated innovations if the signal-to-noise ratio becomes too low, we would expect autocorrelation to show up if the model implied excessive smoothing, which we do not find for our benchmark parametrization with $\lambda = 100$. However, while insignificant at the 5% level (but significant at the 10% level), we do observe that the LM-test statistics for autocorrelation approaches the critical value as λ increases for all but the KE

[17] One major difference is that we use the unfiltered dataseries and model the technical change explicitely, Chirinko and Mallick (2017) smooths the data with a low-pass filter *prior* to the estimation.

	K(E)			KE(L)			KEL(B)			KELB(M)		
λ	σ	NIS	Auto	σ	NIS	Auto	σ	NIS	Auto	σ	NIS	Auto
1	0.00	0.60	2.84	0.27	91.67	9.85	0.11	0.53	2.35	0.00	0.57	0.41
10	0.09	0.77	0.11	0.52	44.86	8.84	0.11	0.72	0.45	0.00	0.78	0.94
50	0.17	0.82	0.00	0.60	0.74	1.00	0.13	0.80	2.25	0.00	0.84	2.02
100	0.19	0.84	0.00	0.69	0.76	0.08	0.16	0.83	2.81	0.00	0.85	2.28
500	0.22	0.86	0.01	0.68	0.83	0.97	0.28	0.86	3.27	0.00	0.88	2.80
1,000	0.24	0.86	0.01	0.71	0.85	1.54	0.35	0.87	3.30	0.00	0.89	3.03
10,000	0.31	0.87	0.00	0.92	0.88	2.22	0.51	0.89	3.41	0.00	0.90	3.45
100,000	0.34	0.89	0.00	0.99	0.89	2.30	0.54	0.89	3.44	0.00	0.91	3.51

Table 5: Private sector: NIS and Breusch-Godfrey test for autocorrelation for different values of λ. NIS is the Normalized Innovations Squared test and Auto is the Breusch-Godfrey test for autocorrelation. The critical values of the NIS is $[0.63; 1.45]$ on a 5% significance level and the critical value of the Breusch-Godfrey test is 3.84 on a 5% significance level (2.71 on a 10% level).

nest, as shown in Table 5. When considering the KEL nest in the manufacturing sector, the LM-test statistics for autocorrelation exceed the critical value on a 5% significance level for all considered values above $\lambda = 100$ (Table 9 in Appendix F). This suggests that a linear trend in the augmenting technical change is generally too restrictive to describe the medium run variation in technical change.[18] On the other hand, if our model implied too little smoothing of the relative technical change (λ small) one might expect to reject the NIS test as the filter can be sensitive to tuning of the measurement noise in particular. If λ is too low, the maximum likelihood estimate of σ_ε^2 will either be too small or that of σ_η^2 be too large, leading to the NIS test being above or below 1, respectively. The consistent performance of the filter with $\lambda = 100$ lends support to both our estimation and identification strategy. Looking at Table 5 we do find the test for well calibrated innovations to be violated for $\lambda = 1$ based on a too low NIS in all nests. Likewise, we also find $\lambda = 10$ to be too low based on the LM-test for autocorrelation in the KEL nest. This indicates that some degree of smoothing of factor augmenting technical change is necessary to obtain a well specified model. Consequently, it can be argued that $\lambda = 50$ or $\lambda = 100$ is an appropriate value of the smoothing parameter, when looking across nests and sectors. Quite encouragingly, these degrees of smoothing do lead to estimates that are not widely different, emphasizing that small changes in the signal-to-noise ratio do not lead to markedly different results.

[18] As comparison to our identification strategy we have estimated (3) by OLS imposing the common linear trend assumption. The estimated elasticity of substitution is almost identical to the elasticity when using $\lambda = 100,000$, no matter the sector or nest considered, confirming that our model nests the common linear trend assumption.

6 Concluding remarks

This paper shows how the Kalman filter can be used to simultaneously provide an estimate of the elasticity of substitution and identify time-varying and potentially biased technical change. By exploiting the natural state space representation of the problem, we avoid a full parametric specification of the structural changes in the economy. Instead, potentially asymmetric growth in the augmenting technical change of production factors is identified by a smoothness restriction. We show in a simulation study that our approach performs well in terms of reproducing the true elasticities and performs better than the common linear trend assumption when technology is non-linear. Using Danish data for the period 1966-2016 and our preferred nest structure, we conclude the following for the aggregate and sectoral level: There is little or moderate substitution between capital and energy as a result of price changes. For the nests containing building capital and materials we find the elasticities to be low or even 0. On the other hand, the elasticities between labor and the capital-energy nest are found to be relatively large, in the range of 0.6-0.9 and not significantly different from unity, hence providing support for a Cobb-Douglas specification. Our results for the augmenting technical change show that labor augmenting technical change has been increasing more than capital augmenting technical change in the long run. However, we find that important "medium-run" fluctuations are present where the augmenting technical change of capital has been increasing more than the augmenting technical change of labor. Our analysis at the sectorial level suggests that this result is driven by periods of relatively slow labor productivity in the service sector.

A Simulation evidence - description of the data generating process

The data series are simulated in accordance with the relative expenditure shares in (2), but an error term $\varepsilon_t \sim N\left(0, \Sigma^{\varepsilon}\right)$ is added. As an example, assume that the first factor of production is capital and the second factor is labor. The prices are simulated according to:

$$r_t = r_0 e^{\gamma_r + log(r_{t-1}) + \varepsilon_t^r}, \quad \varepsilon_t^r \sim N\left(0, \Sigma^r\right) \tag{5}$$

$$w_t = w_0 e^{\gamma_w + log(w_{t-1}) + \varepsilon_t^w}, \quad \varepsilon_t^w \sim N\left(0, \Sigma^w\right) \tag{6}$$

r_t is the usercost of capital and w_t is the real wage rate. The drift parameter $\gamma_r = 0$ is chosen such that the interest rate is a Random Walk, and the parameter γ_w is chosen to reflect an average increase in real wages on 2% per year, similar to what is observed historically. All parameter values are shown in Table 6. We set $r_0 = w_0 = 1$. The variances Σ^r, Σ^w are chosen to match what is observed on average in the data sets used in the empirical application. The augmenting technologies Γ_t^K and Γ_t^L consist of a stochastic and a deterministic part. We vary on the relative importance of these and try three different specifications of the deterministic part:

$$\Gamma_t^K = \Gamma_0^K e^{g(\Gamma^K) + log\left(\Gamma_{t-1}^K\right) + \varepsilon_t^{\Gamma^K}}, \quad \varepsilon_t^{\Gamma^K} \sim N\left(0, \Sigma^{\Gamma^K}\right) \tag{7}$$

$$\Gamma_t^L = \Gamma_0^L e^{g\left(\Gamma^L\right) + log\left(\Gamma_{t-1}^L\right) + \varepsilon_t^{\Gamma^L}}, \quad \varepsilon_t^{\Gamma^L} \sim N\left(0, \Sigma^{\Gamma^L}\right), \tag{8}$$

We assume $\Gamma_0^K = \Gamma_0^L = 1$. The growth rates $g\left(\Gamma^K\right) = \gamma_{\Gamma^K}$ and $g\left(\Gamma^L\right) = \gamma_{\Gamma^L}$ are in the first case are specified such that technology is augmenting labor with a deterministic trend throughout the whole sample period. In the second case, labor is the augmenting factor in the first half of the sample period, whereas capital is the augmenting factor in the second half of the sample period. In the third case we use a Box-Cox transformation of the growth rates given as $g\left(\Gamma^K\right) = \frac{\gamma_{\Gamma^K} t_0}{\lambda_{\Gamma^K}}\left[\left(\frac{t}{t_0}\right)^{\lambda_{\Gamma^K}} - 1\right]$ and $g\left(\Gamma^L\right) = \frac{\gamma_{\Gamma^L} t_0}{\lambda_{\Gamma^L}}\left[\left(\frac{t}{t_0}\right)^{\lambda_{\Gamma^L}} - 1\right]$. γ_{Γ^K} and γ_{Γ^L} are the drift parameters and λ_{Γ^K} and λ_{Γ^L} are the curvature parameters carefully chosen to reflect medium run fluctuations. We set $t_0 = 1$. The variances of the technology parameters $\Sigma^{\Gamma^K}, \Sigma^{\Gamma^L}$ are assumed identical, denoted as Σ^{Γ}, and specified in conjunction with the measurement error variance Σ^{ε} according to:

$$\tilde{\lambda} = \frac{\Sigma^{\varepsilon}}{(\sigma - 1)^2 \Sigma^{\Gamma}} \tag{9}$$

Parameter	Description	Harrod-Neutral	Harrod- and Solow-Neutral	Box-Cox
$\Sigma^{\Delta s}$	Variance of expenditure shares	0.01	0.01	0.01
Σ^{r}	Variance of the interest rate	0.005	0.005	0.005
Σ^{w}	Variance of real wages	0.005	0.005	0.005
γ_r	Drift parameter in the interest rate	0	0	0
γ_w	Drift parameter in the real wage	0.02	0.02	0.02
$\tilde{\lambda}$	Noise-to-signal ratio in the DGP	$1, 10, 100, 1000$	$1, 10, 100, 1000$	$1, 10, 100, 1000$
σ	Elasticity of substitution	$0.2, 0.5, 0.9, 1.3$	$0.2, 0.5, 0.9, 1.3$	$0.2, 0.5, 0.9, 1.3$
γ_{Γ^K}	Drift parameter in Γ^K	0	0 , first half, 0.02 second half	0.01
γ_{Γ^L}	Drift parameter in Γ^L	0.02	0.05 , first half, 0 second half	0.07
λ_{Γ^K}	Curvature parameter in Γ^K	-	-	0.4
λ_{Γ^L}	Curvature parameter in Γ^L	-	-	-0.9

Table 6: Parameter values used in the simulations

$\tilde{\lambda}$ is the inverse of the signal-to-noise ratio in the data generating process. We try different values of this parameter in the simulations. Jointly, Σ^{ε} and Σ^{Γ} are chosen such that we keep the variance in the expenditure shares (called $\Sigma^{\Delta s}$) constant in all simulations to match the observed variance level in the datasets used in the empirical application. The variance Σ^{Γ} is calculated residually as:

$$\Sigma^{\Gamma} = \frac{\Sigma^{\Delta s} - (1-\sigma)^2 (\Sigma^r + \Sigma^w)}{2(\sigma-1)^2 \left(1+\tilde{\lambda}\right)}, \tag{10}$$

B User cost measure

We distinguish between two different types of the usercost. In the short run, we use a measure with installation costs. This usercost is used to estimate the short run elasticity. The long run or "steady state" usercost we define as a usercost in the absence of installation costs. This usercost is used to estimate the long run elasticity. This usercost is very similar to the one used in Statistics Denmark. The measure of the user cost of capital[19] p_t^K is derived by solving the firms maximization problem. The data to calculating the user cost is obtained from Statistics Denmark. The first order condition for investments (Tobin's q) is given by:

$$q_t = p_t^I \left(1 - q_t^{book}\right) + p_t^Y \left(1 - t_t^{corp}\right) * \phi_1 \frac{I_t}{K_{t-1}} \tag{11}$$

q_t is Tobins q, p_t^I is the price of investments in machinery, transportation equipment and inventory and I_t is the quantity. p_t^Y is the price of total production volume and t_t^{corp} is the corporate tax rate. K_t denotes the quantity of capital. ϕ_1 measures the marginal effect of installation costs. Thus, higher values of ϕ_1 lead to larger short run fluctuations in the user cost. In the steady state user cost this is set to 0, whereas it is set to 2.5 in the user cost with adaptive expectations. q_t^{book}

[19]The usercost of buildings is derived in the same way.

is the shadow value of the book value of capital:

$$q_t^{book} = \frac{1}{1+r_{t+1}^K}\left(\left(1-\delta^{book}\right)q_{t+1}^{book} + t_{t+1}^{corp}\delta^{book}\right) \tag{12}$$

r_{t+1}^K is the interest rate, which is measured as the banks effective lending rate plus a risk premium on 2%. δ^{book} is the depreciation rate of book capital, which is set to 15%. The terminal condition is given as:

$$q_T^{book} = \frac{t_t^{corp}\delta^{book}}{r_t^K + \delta^{book}} \tag{13}$$

The user cost is then given as:

$$\begin{aligned} p_t^K = \quad & q_t\left(1+r_{t+1}^K\right) - (1-\delta_{t+1})q_t \\ & -t_{t+1}^{corp} * r_{t+1}^K * D^{firm} p_t^I \\ & -\tfrac{1-t_t^{corp}}{1+t_t^Y} p_{t+1}^Y \left(1-t_{t+1}^{corp}\right) * \Phi_{t+1}^K \end{aligned} \tag{14}$$

δ_t is the depreciation rate of capital, which can be calibrated as shown below. D^{firm} is the debt share of firms, which is set to 60%. Φ_{t+1}^K is the quadratic installation costs given by:

$$\Phi_{t+1}^K = \frac{\phi_1}{2}K_t\left(\frac{I_{t+1}}{K_t}\right)^2 \tag{15}$$

$$\delta_t = \frac{K_t + I_t}{K_{t-1}} - 1 \tag{16}$$

C Data

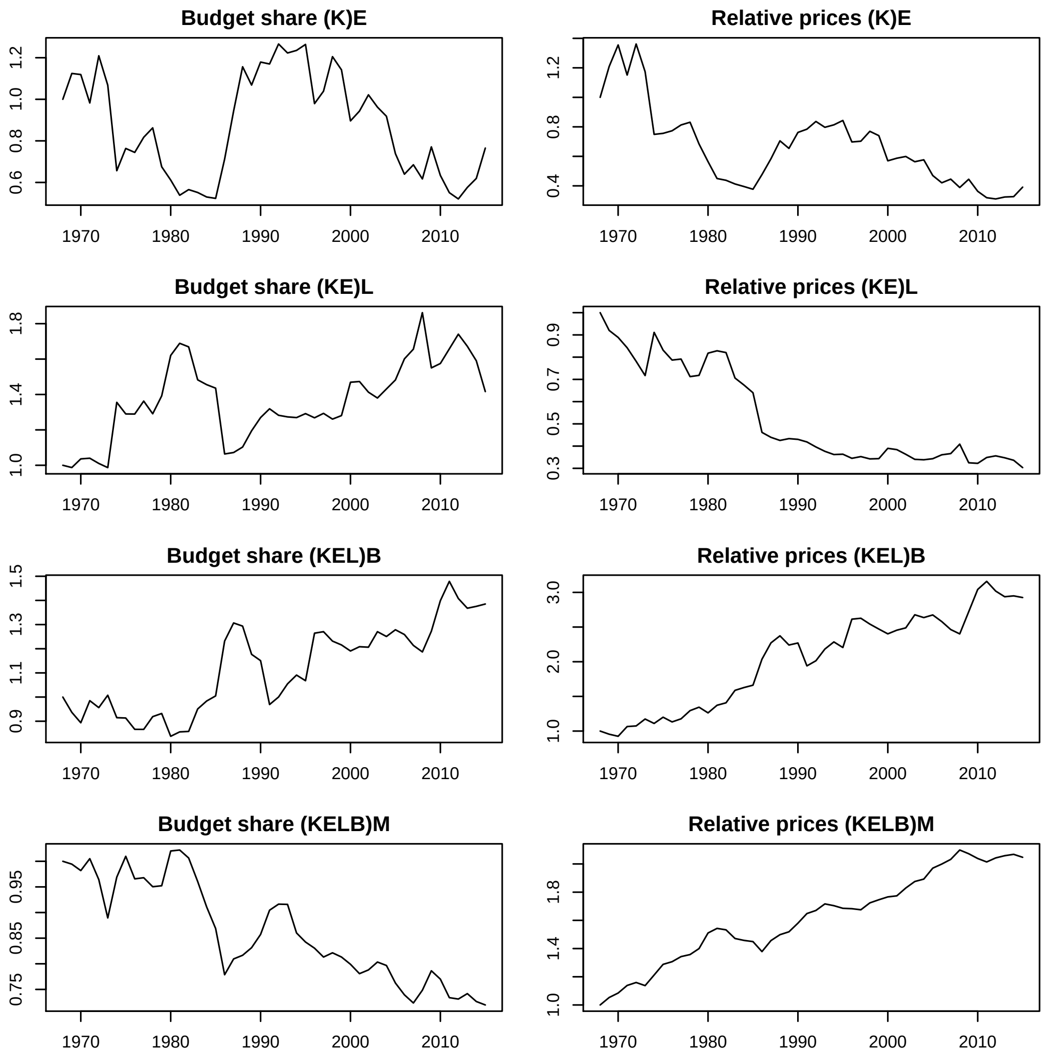

Figure 3: Private sector: Relative prices and budget shares. Paranthesis indicates the nested factors.

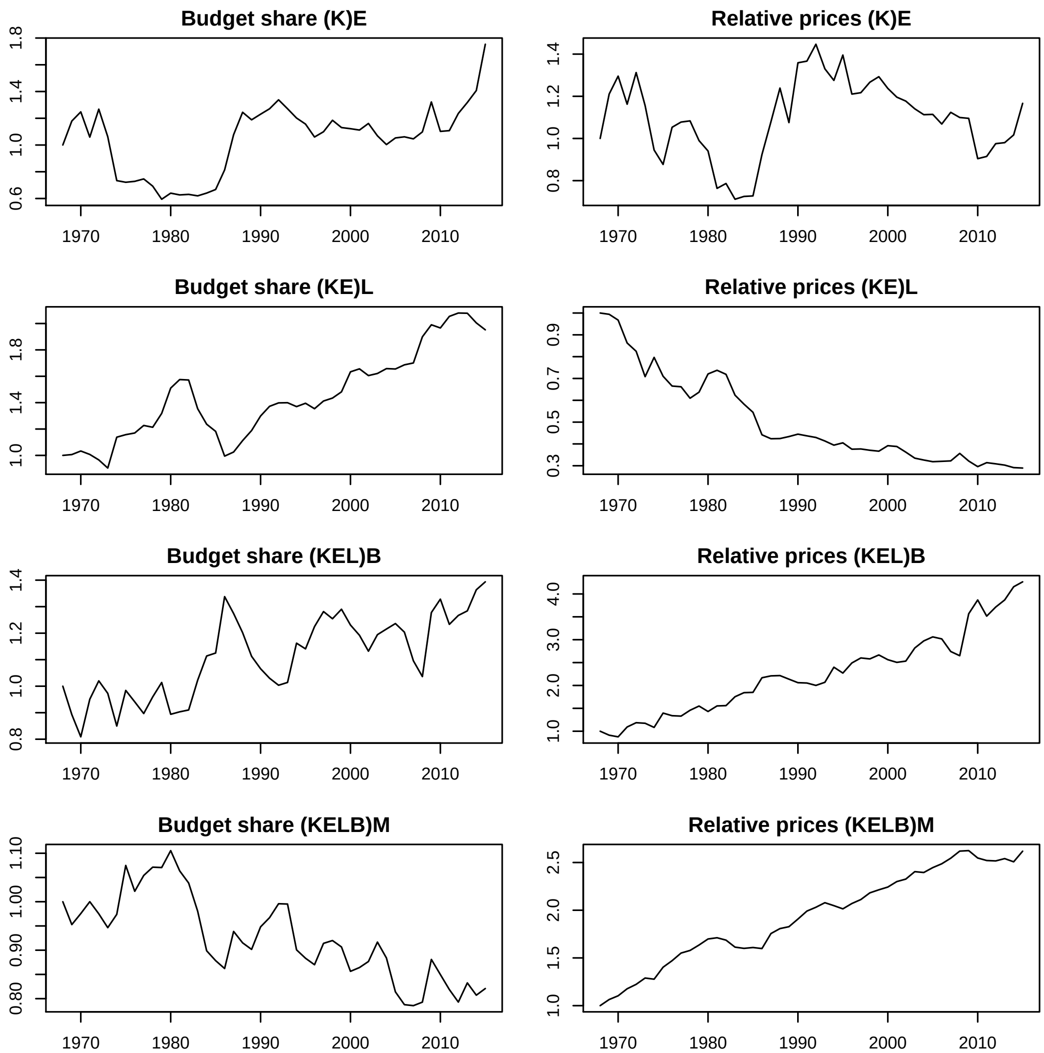

Figure 4: Manufacturing: Relative prices and budget shares. Paranthesis indicates the nested factors.

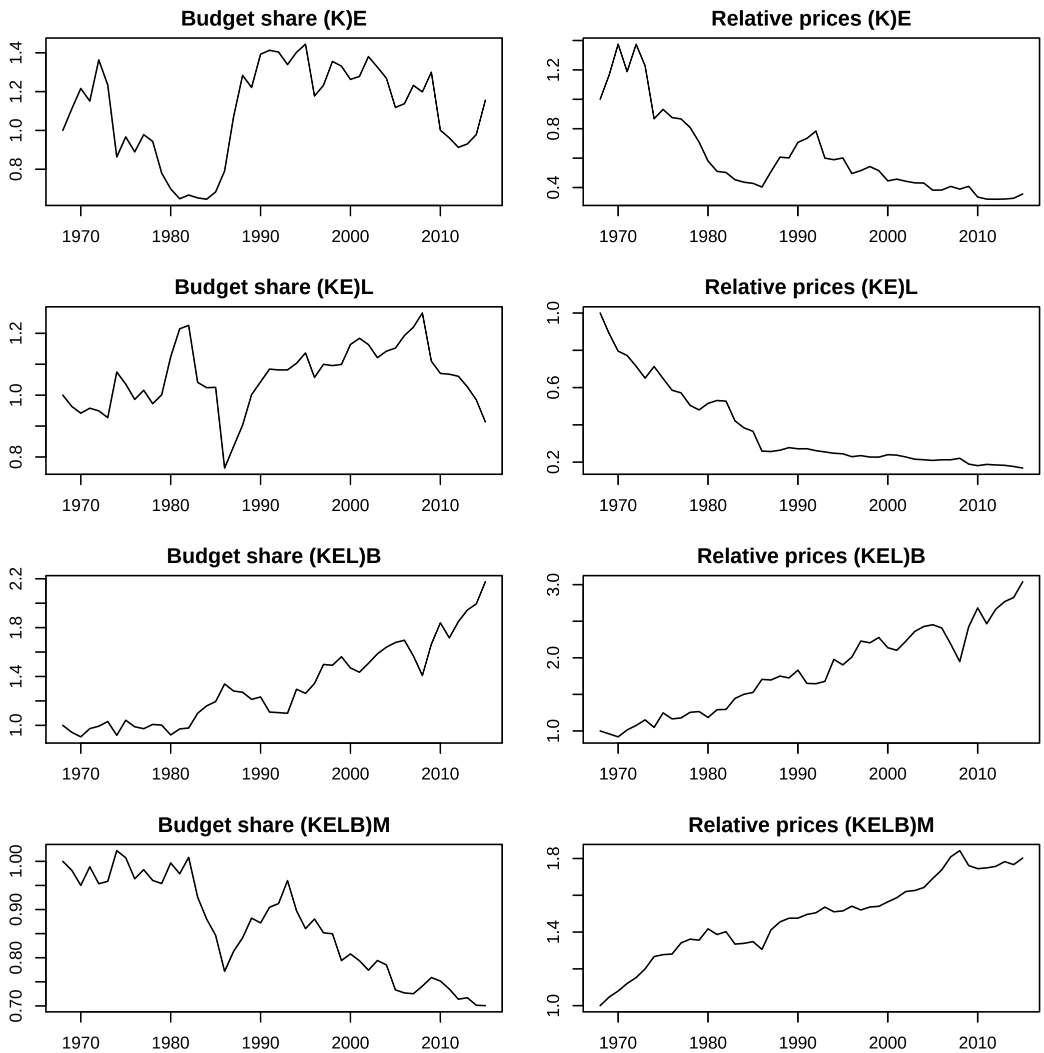

Figure 5: Private service: Relative prices and budget shares. Paranthesis indicates the nested factors.

D Estimation results in manufacturing and private services

	(K)E	(KE)L	(KEL)B	(KELB)M
σ	0.18	0.87	0.00	0.09
	(−0.39;1.15)	(0.10;1.68)	(−0.59;0.77)	(−0.31;0.50)
α	−0.36	−0.16	−0.24	−0.62
	(−0.42;−0.10)	(−0.29;−0.07)	(−0.36;−0.07)	(−0.81;−0.31)
k	0	0	0	0
Likelihood	72.99	105.87	110.21	106.72
R^2	0.72	0.82	0.89	0.57
Autocorrelation	[0.46]	[0.09]	[0.76]	[0.61]
Heteroskedasticity	[0.78]	[0.32]	[0.00]	[0.16]
Normality	[0.92]	[0.01]	[0.99]	[0.66]
NIS	0.79	0.82	0.79	0.84

Table 7: Manufacturing: Estimated results in a (((KE)L)B)M nest structure. Terms in parenthesis are respectively the lower and upper confidence intervals on a 5% significance level. Terms in brackets are p-values for the misspecification tests. The critical values of the Normalized Innovations Squared test (NIS) are $[0.63; 1.45]$ on a 5% significance level.

	(K)E	(KE)L	(KEL)B	(KELB)M
σ	0.36	0.61	0.00	0.00
	(0.05;0.97)	(0.36;1.59)	(−0.61;1.08)	(−0.68;0.69)
α	−0.42	−0.16	−0.10	−0.43
	(−0.55;−0.14)	(−0.21;−0.05)	(−0.23;0.01)	(−0.58;−0.19)
k	0	0	0	0
Likelihood	84.20	136.61	119.48	114.98
R^2	0.85	0.97	0.96	0.55
Autocorrelation	[0.97]	[0.18]	[0.11]	[0.53]
Heteroskedasticity	[0.41]	[0.34]	[0.00]	[0.59]
Normality	[0.14]	[0.06]	[0.95]	[0.99]
NIS	0.81	0.80	0.72	0.84

Table 8: Private services: Estimated results in a (((KE)L)B)M nest structure. Terms in parenthesis are respectively the lower and upper confidence intervals on a 5% significance level. Terms in brackets are p-values for the misspecification tests. The critical values of the Normalized Innovations Squared test (NIS) are $[0.63; 1.45]$ on a 5% significance level.

E Augmenting technical change in manufacturing and private services

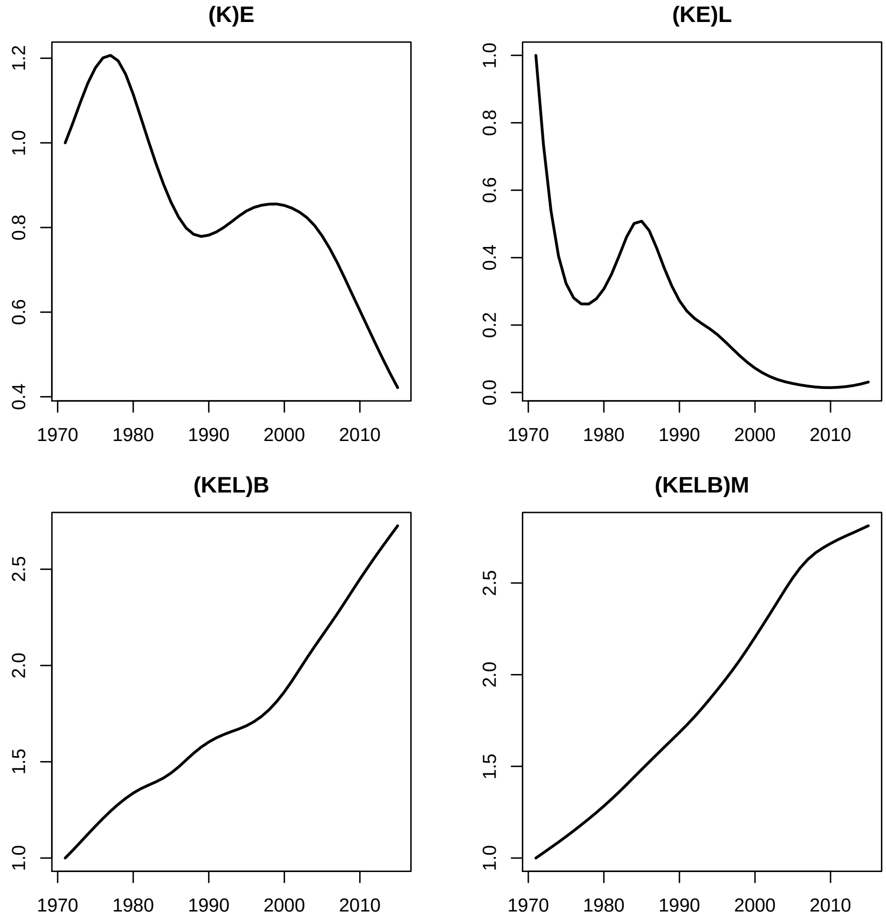

Figure 6: Manufacturing: Relative augmenting technical change in a (((KE)L)B)M nest structure. Paranthesis indicates the nested factors. That is, the graph shows the relative augmenting technologies of e.g. capital relative to energy. Index year is 1971.

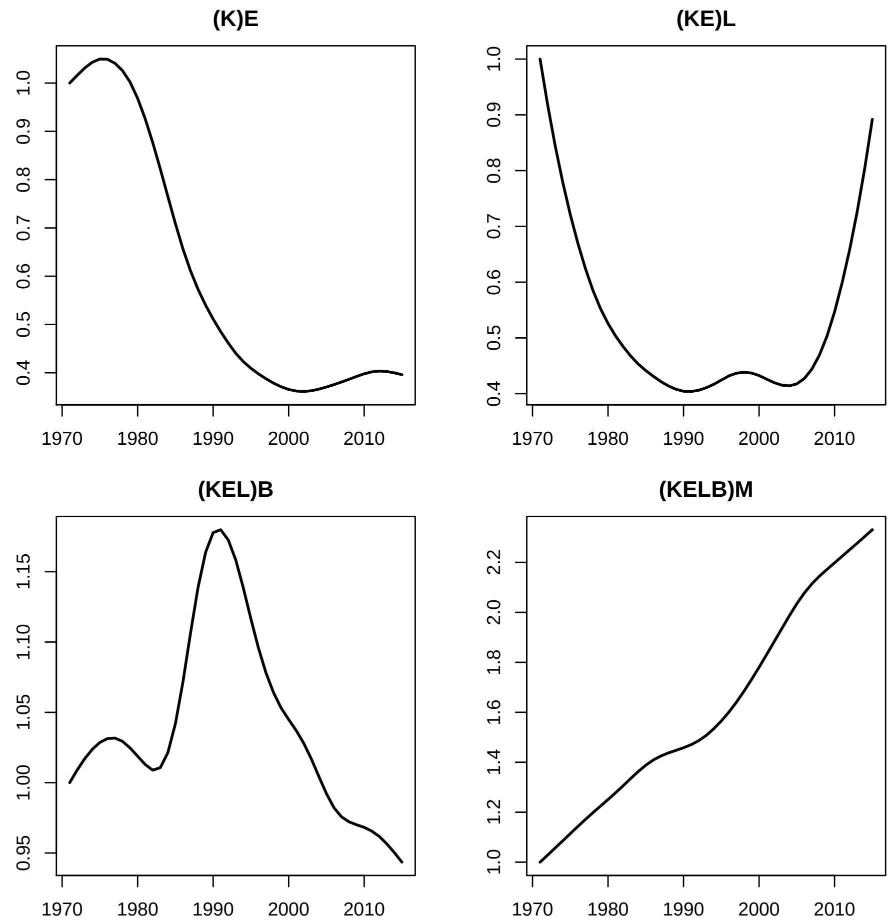

Figure 7: Private services: Relative augmenting technical change in a (((KE)L)B)M nest structure. Paranthesis indicates the nested factors. That is, the graph shows the relative augmenting technologies of e.g. capital relative to energy. Index year is 1971.

F Filter performance, consistency and the identification strategy

To evaluate filter performance we would like to know if the filtered state is a reasonably prediction of the true value. However, as the true state is unknown filter consistency is usually based on information on the innovations in the observation equation. If the filter is consistent, the standardized forecast errors will be a zero-mean and homoskedastic white noise process. This can be evaluated either by graphically inspecting the standardized innovations or (more formally) by considering the Normalized Innovations Squared test (NIS). The NIS test has the following test statistic:

$$m_t = \varepsilon_t^T F_t^{-1} \varepsilon_t, \tag{17}$$

	K(E)			KE(L)			KEL(B)			KELB(M)		
λ	σ	NIS	Auto	σ	NIS	Auto	σ	NIS	Auto	σ	NIS	Auto
1	0.09	0.61	1.92	0.06	0.59	0.87	0.04	0.49	0.00	0.21	0.61	0.86
10	0.13	0.71	1.07	0.19	0.71	0.18	0.02	0.66	0.02	0.06	0.80	0.03
50	0.16	0.75	0.74	0.62	0.79	2.01	0.00	0.76	0.06	0.07	0.83	0.18
100	0.18	0.79	0.55	0.87	0.82	2.79	0.00	0.79	0.09	0.09	0.84	0.25
500	0.33	0.85	0.33	0.67	6.68	0.09	0.01	0.81	0.23	0.16	0.85	0.50
1,000	0.43	0.86	0.28	0.69	6.59	0.01	0.02	0.81	0.27	0.19	0.86	0.62
10,000	0.68	0.88	0.17	0.62	6.62	0.06	0.02	0.82	0.35	0.33	0.88	0.96
100,000	0.72	0.89	0.15	0.58	6.66	0.09	0.00	0.82	0.37	0.40	0.89	1.15

Table 9: Manufacturing: NIS and Breusch-Godfrey test for autocorrelation for different values of λ. NIS is the Normalized Innovations Squared test and Auto is the Bresuch Godfrey test for autocorrelation. The critical values of the NIS is $[0.63; 1.45]$ on a 5% significance level and the critical value of the Breusch-Godfrey test is 3.84 on a 5% significance level.

	K(E)			KE(L)			KEL(B)			KELB(M)		
λ	σ	NIS	Auto	σ	NIS	Auto	σ	NIS	Auto	σ	NIS	Auto
1	0.27	0.57	3.54	0.51	74.46	9.10	0.00	0.47	2.72	0.00	0.56	2.89
10	0.35	0.76	0.09	0.39	38.33	3.33	0.00	0.59	2.70	0.00	0.76	0.08
50	0.36	0.80	0.00	0.49	0.79	2.49	0.00	0.70	2.79	0.00	0.82	0.16
100	0.36	0.81	0.00	0.61	0.80	1.79	0.00	0.72	2.62	0.00	0.84	0.39
500	0.32	0.83	0.05	0.93	0.83	0.46	0.01	0.77	2.76	0.00	0.87	1.17
1,000	0.33	0.84	0.08	1.14	0.85	0.19	0.05	0.78	2.76	0.00	0.89	1.48
10,000	0.62	0.87	0.12	1.82	0.88	0.01	0.30	0.80	2.42	0.00	0.91	1.94
100,000	0.91	0.89	0.15	1.99	0.89	0.00	0.39	0.80	2.20	0.00	0.91	2.02

Table 10: Private services: NIS and Breusch-Godfrey test for autocorrelation for different values of λ. NIS is the Normalized Innovations Squared test and Auto is the Bresuch Godfrey test for autocorrelation. The critical values of the NIS is $[0.63; 1.45]$ on a 5% significance level and the critical value of the Breusch-Godfrey test is 3.84 on a 5% significance level.

where F_t is the covariance matrix of the innovations. If the assumptions are correct, m_t will be $\chi^2(1)$ distributed, implying that the T period moving average, $\bar{m}_T$, has a $T\chi^2(T)$ distribution (applying the ergodic property of the innovations). Hence, the null hypothesis is $E[m] = 1$ and can be tested by computing the moving average of (17) recursively for an increasing sample size and compare the test statistics to the critical values (see Figure 8-10).

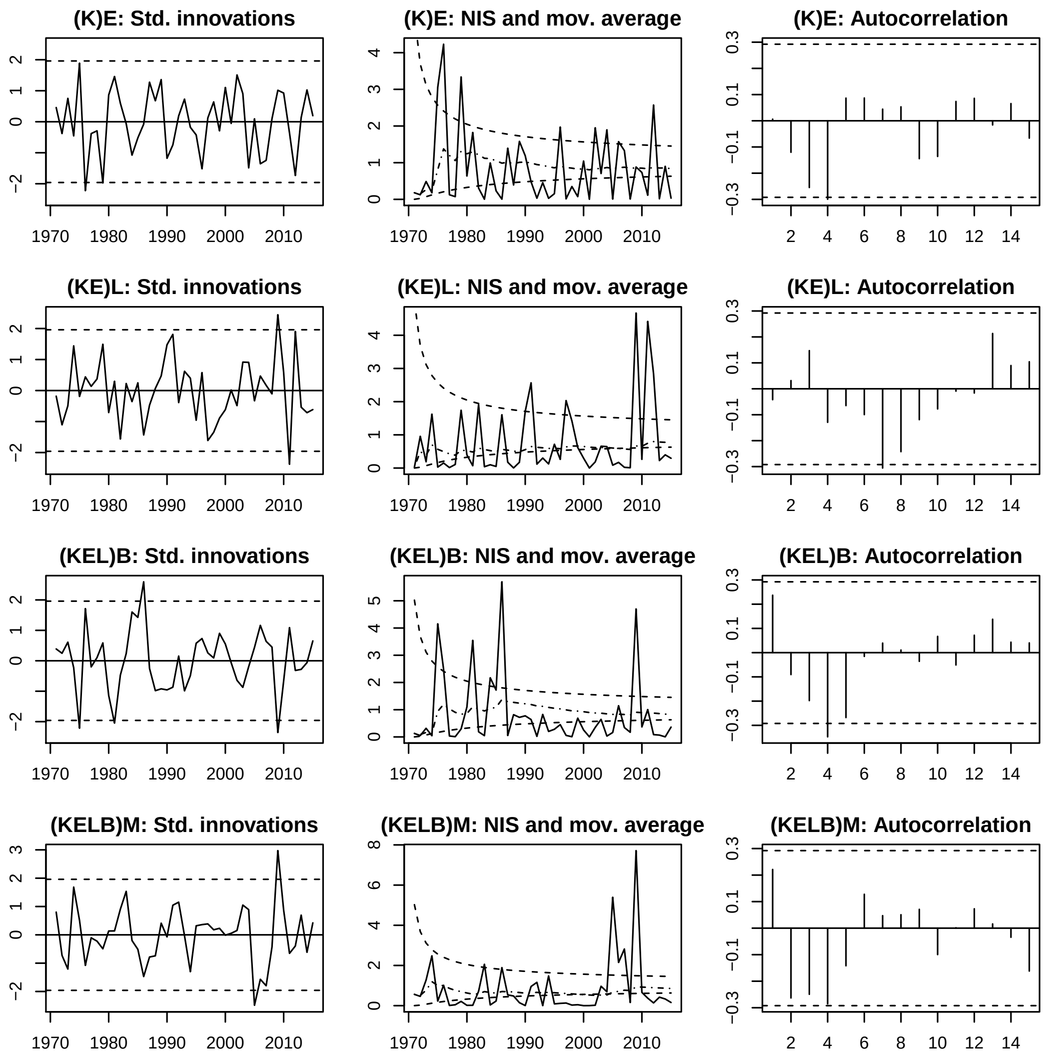

Figure 8: Private sector: Diagnostics where a (((KE)L)B)M structure is applied. Paranthesis indicates the nested factors.

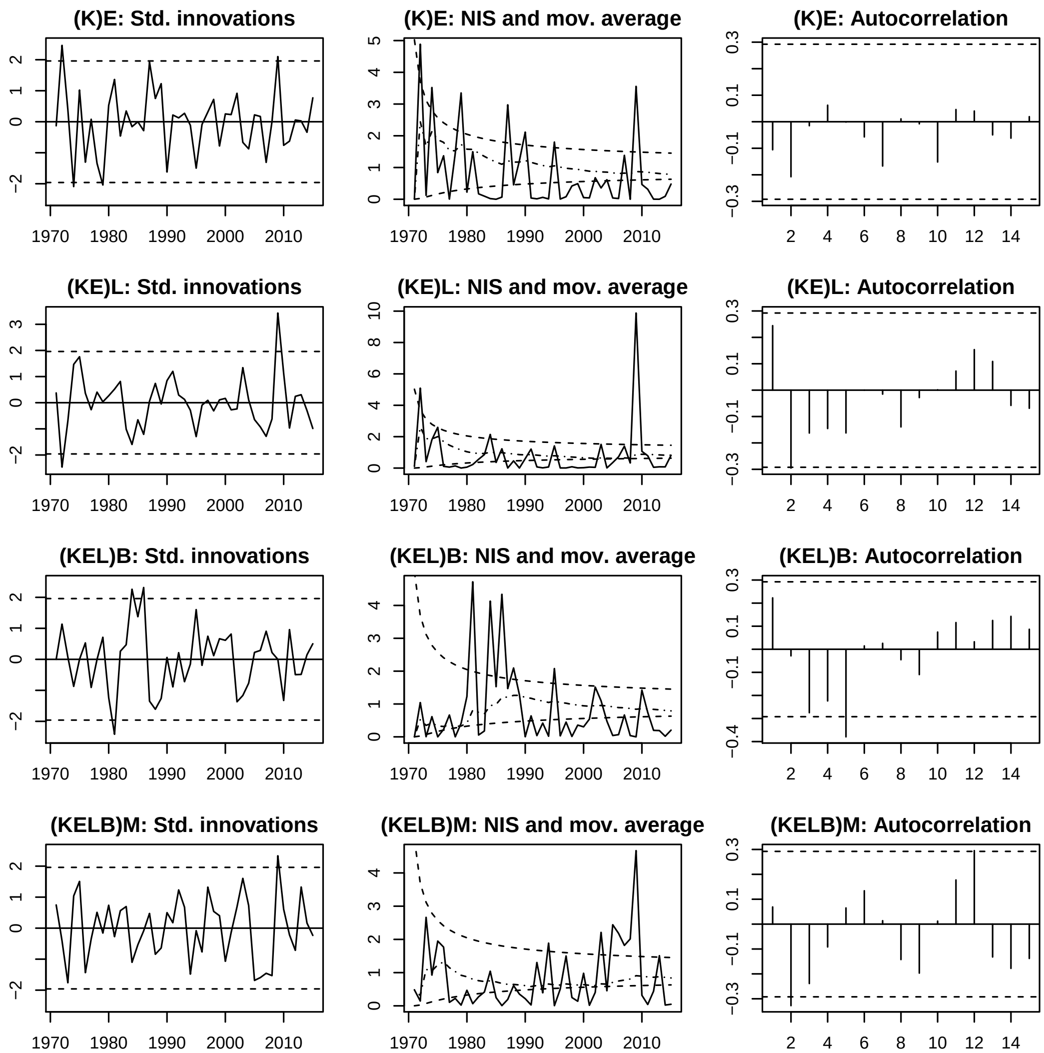

Figure 9: Manufacturing: Diagnostics where a (((KE)L)B)M structure is applied. Paranthesis indicates the nested factors.

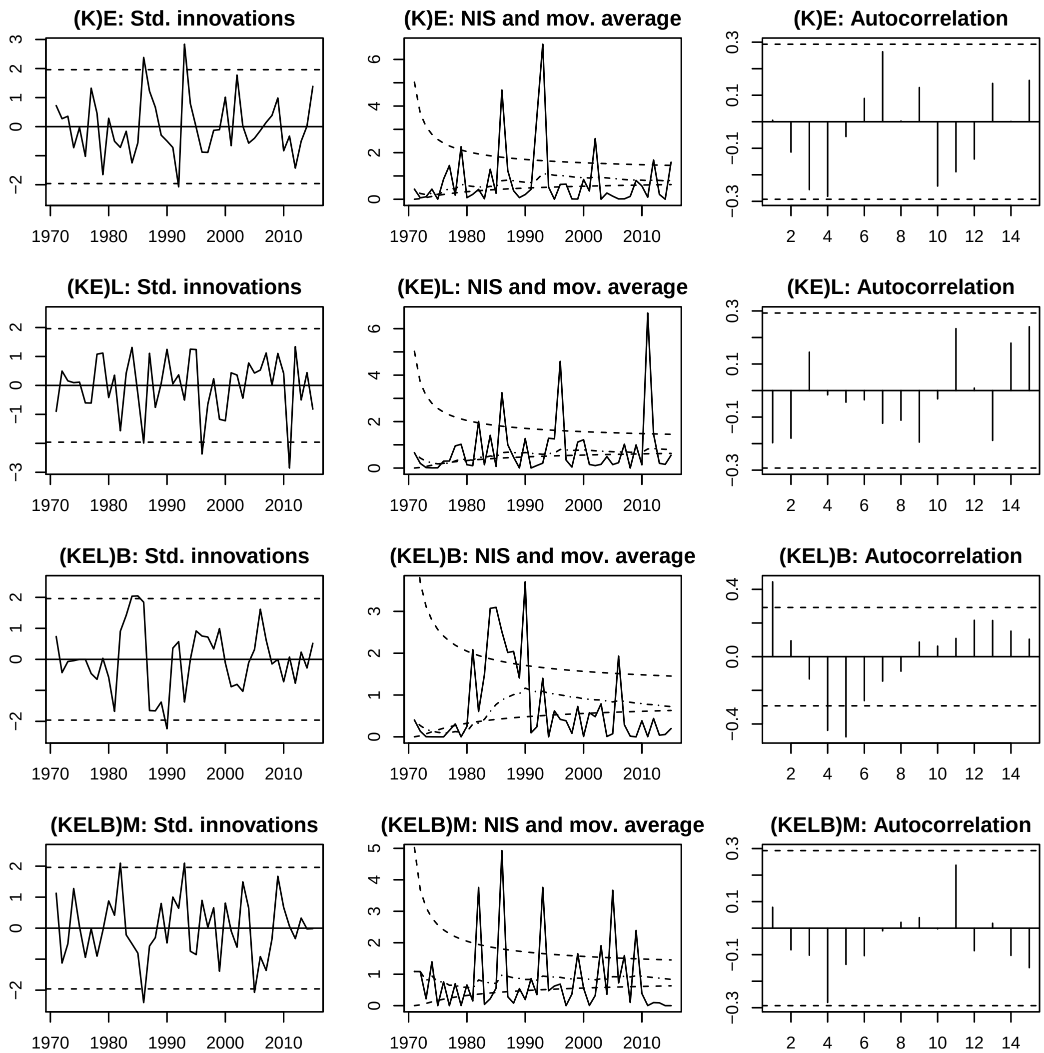

Figure 10: Private services: Diagnostics where a (((KE)L)B)M structure is applied. Paranthesis indicates the nested factors.

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